Topology II, lecture 15–16

By the Prop.: $\exists \beta \in H^{n-k}(M;\mathbb{Z})$ s.t. $U_M^*(\beta) = \varphi$.

$$U_M^*(\beta) = \langle - \cup \beta, [M] \rangle = \varphi(-) \quad \Rightarrow \quad \langle \alpha \cup \beta, [M] \rangle = \varphi(\alpha) = 1.$$

$$H^n(M;\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_n(M;\mathbb{Z}), \mathbb{Z})$$ Recall that the torsion of $H_{n-1}(M;\mathbb{Z})$ is 0.

$$\alpha \cup \beta \longmapsto \underbrace{\langle \alpha \cup \beta, - \rangle}_{\text{this is a generator}} \Rightarrow \alpha\cup\beta \text{ is a generator}$$

Let $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (but not $\mathbb{O}$)

$$FP^n := F^{n+1} \setminus \{0\} \big/ (x_0,\dots,x_n) \sim (y_0,\dots,y_n) \Leftrightarrow \exists \lambda \in F^\times \text{ s.t. } \lambda x_i = y_i \ \forall i = 0,\dots,n$$ projective space

The topology is the quotient topology from $F^{n+1} \setminus \{0\}$

FP^n are n-dimensional cpt connected manifolds

$U_i \subseteq FP^n \qquad U_i = \{[x_0,\dots,x_n] \in FP^n \mid x_i \neq 0\}, \qquad F^n \xrightarrow{\cong} U_i$

$$(x_1,\dots,x_n) \longmapsto [x_1,\dots,x_{i-1},1,x_{i},\dots,x_n]$$

$FP^n = U_0 \cup \dots \cup U_n$

Let $S(F^{n+1})$ be the unit sphere in F^{n+1}. Then the projection

$S(F^{n+1}) \twoheadrightarrow FP^n$ is surjective: $[x_0,\dots,x_n] = \left[\frac{x_0}{\|x\|},\dots,\frac{x_n}{\|x_i\|}\right]$ where $\|x\| = \sqrt{\sum x_i^2}$

$\underset{\sim\sim}{F=\mathbb{R}}$: $\mathbb{R}P^n$ is a CW complex

$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & \mathbb{R}P^n \arrow[d] \\ D^n \arrow[r] & \mathbb{R}P^{n+1} \end{tikzcd}$$

$$H_n(\mathbb{R}P^n;\mathbb{Z}) = \begin{cases} 0 & n \geq 2,\ 2 \mid n \\ \mathbb{Z} & 2 \nmid n \end{cases} \quad \Rightarrow \quad \mathbb{R}P^n \text{ is orientable if } 2\nmid n \text{ and non-orientable if } 2 \mid n.$$

$$H_i(\mathbb{R}P^n, \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases} \quad \Rightarrow \quad H^i(\mathbb{R}P^n, \mathbb{F}_2) = \begin{cases} 0 & i > n \\ \mathbb{F}_2 & 0 \leq i \leq n \end{cases}$$

$F = \mathbb{C}$:

$$\begin{tikzcd} S^{2n+1} \arrow[r] \arrow[d] & \mathbb{C}P^n \arrow[d] \\ D^{2n+2} \arrow[r] & \mathbb{C}P^{n+1} \end{tikzcd}$$

CW complex with no odd dimensional cells

$$H_i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \le i \le 2n,\ 2 \mid i \\ 0 & \text{else} \end{cases}$$

$$\text{UCT} \Rightarrow H^i(\mathbb{C}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \le i \le 2n,\ 2 \mid i \\ 0 & \text{else} \end{cases}$$

$F = \mathbb{H}$:

$$\begin{tikzcd} S^{4n+3} \arrow[r] \arrow[d] & \mathbb{H}P^n \arrow[d] \\ D^{4n+4} \arrow[r] & \mathbb{H}P^{n+1} \end{tikzcd}$$

$$H_i(\mathbb{H}P^n; \mathbb{Z}) \cong H^i(\mathbb{H}P^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & 0 \le i \le 4n,\ 4 \mid i \\ 0 & \text{else} \end{cases}$$

What about $\mathbb{O}$? The projective spaces $\mathbb{O}P^1 = S^8$ and $\mathbb{O}P^2$ (Cayley plane) can be defined. The relation won't be an eq. relation. Of course it can be made into one but it won't yield a manifold.

Next goal: compute the cohomology rings using the Cor. (p. 72).

11.06.2018

Thm. 1) $H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha_n]/(\alpha_n^{n+1})$ where $|\alpha_n| = 1$

$H^*(\mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[\alpha]$ where $|\alpha| = 1$

$\quad \cong$

$H^*(BC_2; \mathbb{F}_2) \cong \mathrm{Ext}^*_{\mathbb{F}_2[C_2]}(\mathbb{F}_2, \mathbb{F}_2)$ } this was shown in the exercises

2) $H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha_n]/(\alpha_n^{n+1})$ where $|\alpha_n| = 2$

$H^*(\mathbb{C}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$ where $|\alpha| = 2$

3) $H^*(\mathbb{H}P^n; \mathbb{Z}) \cong \mathbb{Z}[\alpha_n]/(\alpha_n^{n+1})$ where $|\alpha_n| = 4$

$H^*(\mathbb{H}P^\infty; \mathbb{Z}) \cong \mathbb{Z}[\alpha]$ where $|\alpha| = 4$

We will only prove 2), the rest is quite similar.

Before the proof, recall the construction of $-P^\infty$:

$\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$ and this is the same as the map in

$[x_0, \dots, x_n] \longmapsto [x_0, \dots, x_n, 0]$ the pushout diagram above

Note that all the $\mathbb{CP}^n$ are compact.

$$\mathbb{CP}^\infty = \bigcup_{n \geq 0} \mathbb{CP}^n$$ is a CW complex by the topology induced by the $\mathbb{CP}^n$s.

(see Topology 1)

PF OF THM: Induction on n.

$n=1$: $\mathbb{CP}^1 \cong S^2 = \mathbb{C} \cup \{\infty\}$ Riemann sphere

$[z_0, z_1] \longmapsto \frac{z_0}{z_1}$

$H^*(\mathbb{CP}^1; \mathbb{Z}) \cong H^*(S^2; \mathbb{Z}) \cong \mathbb{Z}[\alpha_1]/(\alpha_1^2)$ where $|\alpha_1| = 2$, we have seen this before.

Ind. step. $H^2(\mathbb{CP}^n, \mathbb{Z}) \xrightarrow{\iota_n^*} H^2(\mathbb{CP}^1; \mathbb{Z})$ where $\iota_n : \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^n$

$[z_0, z_1] \longmapsto [z_0, z_1, 0, \dots, 0]$

ι_n^* is an isomorphism by cellular (co)homology

$\Rightarrow \exists! \, \alpha_n$ s.t. $\iota_n^*(\alpha_n) = \alpha_1$.

Suppose the statement holds for $n-1$, i.e. $H^*(\mathbb{CP}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}[\alpha_{n-1}]/\alpha_{n-1}^2$ $|\alpha_{n-1}| = 2$

By the induction assumption, $\alpha_{n-1}^i \in H^{2i}(\mathbb{CP}^{n-1}, \mathbb{Z}) \cong \mathbb{Z}$ for $i \leq n-1$ is a generator.

Again using cellular (co)homology it follows that

$$H^\ell(\mathbb{CP}^n; \mathbb{Z}) \xrightarrow{\iota^*} H^\ell(\mathbb{CP}^{n-1}; \mathbb{Z})$$

is an iso for $0 \leq \ell \leq 2n-1$.

As α_n and α_{n-1} both get mapped to α_1, $\iota^*(\alpha_n) = \alpha_{n-1}$.

$\Rightarrow$ $0 \leq i \leq 2n-1$: $\iota^*(\alpha_n^i) = (\iota^*(\alpha_n))^i = \alpha_{n-1}^i$

$\Rightarrow$ $\alpha_n^i \in H^{2i}(\mathbb{CP}^n; \mathbb{Z}) \cong \mathbb{Z}$ is a generator for $0 \leq i \leq n-1$

$\mathbb{Z}[\bar{\alpha}_n] \longrightarrow H^*(\mathbb{CP}^n; \mathbb{Z})$ where $\bar{\alpha}_n$ is a formal variable, $|\bar{\alpha}_n| = 2$.

$H^{2n+2}(\mathbb{CP}^n; \mathbb{Z}) = 0$, $\alpha_n^{n+1} = 0$ in $H^*(\mathbb{CP}^n; \mathbb{Z})$

$$\begin{tikzcd}
\mathbb{Z}[\bar{\alpha}_n] \arrow[rr] \arrow[dr] & & H^*(\mathbb{CP}^n; \mathbb{Z}) \\
& \mathbb{Z}[\alpha_n]/(\alpha_n^{n+1}) \arrow[ur, "\exists! \varphi"'] &
\end{tikzcd}$$

$\alpha_n \longmapsto \alpha_n$

NTS: $\alpha_n^n \in \underbrace{H^{2n}(\mathbb{CP}^n, \mathbb{Z})}_{\cong \mathbb{Z}}$ is a generator

By the corollary of PD about the intersection form

$$H^k(M)_{free} \otimes H^{n-k}(M)_{free} \xrightarrow{\cup_M} \mathbb{Z} \cong H^n(M)$$

and the already proven facts that

$$\alpha_n^{n-1} \in H^{2n-2}(\mathbb{C}P^n;\mathbb{Z}) \quad \text{and} \quad \alpha_n \in H^2(\mathbb{C}P^n;\mathbb{Z}) \quad \text{are generators}$$

we have that $\exists m$ s.t. $\alpha_n \cup m\cdot\alpha_n^{n-1} \in H^{2n}(\mathbb{C}P^n;\mathbb{Z})$ is a generator

But $\underbrace{\alpha_n \cup m\cdot\alpha_n^{n-1}}_{=m\cdot\alpha_n^n}$ being a generator implies $m=\pm 1$, hence α_n^n is also a generator.

Now for the assertion about $\mathbb{C}P^\infty$:

$H^\ell(\mathbb{C}P^\infty;\mathbb{Z}) \xrightarrow{\iota^*} H^\ell(\mathbb{C}P^n;\mathbb{Z})$ is an iso for $0\le \ell \le 2n$ where $\iota: \mathbb{C}P^n \hookrightarrow \mathbb{C}P^\infty$ is the canonical inclusion. $\square$

Application. $\mathbb{C}P^1 \cong S^2$.

$\mathbb{C}P^2 \not\cong S^4$ or $S^2\times S^2$ because they have different homology (additively)

$\mathbb{C}P^3 \overset{?}{\cong} S^2\times S^4$? Künneth or cellular chains:

$$H^\ell(S^2\times S^4;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \ell = 0,2,4,6 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ cohomology does not help, neither does homology.

π_1 is also unhelpful: both are simply connected

π_n might help but they are hard to compute.

But the ring structure does help:

$$H^*(\mathbb{C}P^3;\mathbb{Z}) \cong \mathbb{Z}[\alpha_3]/\alpha_3^4 \qquad |\alpha_3| = 2$$

$$H^*(S^2\times S^4;\mathbb{Z}) \underset{\text{Künneth}}{\cong} H^*(S^2;\mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^4;\mathbb{Z}) \cong \mathbb{Z}[\beta]/\beta^2 \otimes \mathbb{Z}[\gamma]/\gamma^2 \qquad \begin{matrix}|\beta|=2\\ |\gamma|=4\end{matrix}$$

$$\cong \mathbb{Z}[\beta,\gamma]/(\beta^2,\gamma^2)$$

Since $\alpha_3^2 \neq 0$, this indeed distinguish between them: as $\cup$ is htp invariant, we have $\mathbb{C}P^3 \not\cong S^2\times S^4$.

$\mathbb{H}P^1 \cong S^4$

$\mathbb{H}P^2 \not\cong S^4$ or $S^2\times S^2$

Again using chg rings: $\mathbb{H}P^3 \not\cong S^4 \times S^8$.

This is an important tool for classifying manifolds.

Manifolds with boundary

Def. An **n-dim manifold with boundary** is a T2 space M s.t. $\forall x \in M$ $\exists U$ open nbhd. for which $U \cong \mathbb{R}^n$, or $\mathbb{R}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$.

Suppose that M is a mf with bdry, $x \in M$, $x \in U \cong \mathbb{R}^n$, where x is sent to 0. Excision: $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z}) \cong \mathbb{Z}$

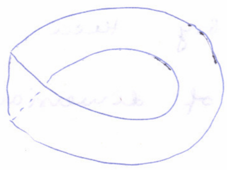
On the other hand, if $x \in U \cong \mathbb{R}^n_+$ with x corresponding to $\bar{x} \in \partial \mathbb{R}^n_+ = \{(x_1, \dots, x_{n-1}, 0) \in \mathbb{R}^n_+\}$ then by excision we have $H_n(M, M \setminus \{x\}; \mathbb{Z}) \cong H_n(\mathbb{R}^n_+, \mathbb{R}^n_+ \setminus \{\bar{x}\}; \mathbb{Z}) = 0$ since $\mathbb{R}^n_+$ and $\mathbb{R}^n_+ \setminus \{x\}$ are both contractible.

Def. For an n-manifold with boundary M we set
$\partial M := \{x \in M \mid H_n(M, M \setminus x; \mathbb{Z}) = 0\}$.

Of course we could have defined ∂M with charts as well.

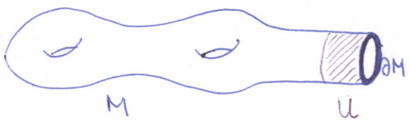
Ex. $\partial D^n = S^{n-1}$, $\partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$

$M =$ Möbius band $= I \times I / (0,t) \sim (1, 1-t)$


Möbius band

$\Rightarrow \partial M = S^1$

$Cyl = I \times I / (0,t) \sim (1,t)$ $\qquad \partial Cyl = S^1 \amalg S^1$



Def. M an n-mf. w/ bdy. A collar neighbourhood of ∂M is an open $U \subseteq M$ s.t. $\partial M \subseteq U$ and $U \cong \partial M \times [0,1)$ where ∂M corresponds to $\partial M \times \{0\}$.

Prop. M cpt n-dim mf w/ bdy. Then there is a collar nbhd of ∂M.

Pf (SKETCH): $M' := M \cup_{\partial M} \partial M \times [0,1]$ pushout where $x \sim (x,0)$ for $x \in \partial M$.

Goal: construct a homeomorphism $M \xrightarrow[\cong]{h} M'$.

Since M is compact, ∂M is also compact.

There exist functions $\varphi_1, \dots, \varphi_\ell : \partial M \longrightarrow [0,1]$ continuous such that $\{V_i := \varphi_i^{-1}(0,1] \mid i = 1, \dots, \ell\}$ cover ∂M, $\overline{V_i} \subseteq U_i \cong \mathbb{R}^n_+$ in M and $\sum \varphi_i = 1$. (Partition of unity)

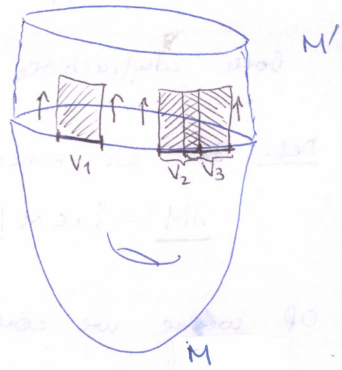
This can be proven by using bump functions, also works for para-compact spaces.

$\Rightarrow \quad 0 \le \psi_k := \varphi_1 + \dots + \varphi_k \le 1 \quad \forall k = 1, \dots, \ell$

$M' \supseteq M_k := M \cup \{(x,t) \in \partial M \times [0,1] \mid t \le \psi_k(x)\}$

There is a chain of homeomorphisms

$$M = M_0 \xrightarrow{\cong} M_1 \xrightarrow{\cong} M_2 \xrightarrow{\cong} \dots \xrightarrow{\cong} M_{\ell-1} \xrightarrow{\cong} M_\ell = M'$$



□

If M is an n-mf w/ bdy then $M \setminus \partial M$ is a typically non-compact manifold w/o boundary of dimension n.

Def. An n-manifold w/ boundary is <u>R-orientable</u> if $M \setminus \partial M$ is R-orientable as a manifold w/o boundary. As before, $\mathbb{Z}$-orientable is called orientable.

Ex. Möbius band: not orientable

Cylinder, D^n are orientable.

Let M be a cpt n-mf w/ bdy. Then a collar exists and

$M = M \setminus \partial M \cup \underbrace{\partial M \times [0,\varepsilon)}_{\text{collar nbhd.}}$

$H_i(M, \partial M; R) \cong H_i(M, \partial M \times [0,\varepsilon); R) \underset{\text{exc.}}{\cong} H_i(M \setminus \partial M, \partial M \times (0,\varepsilon); R)$

M is cpt. $\Rightarrow$ the complement of $\partial M \times (0,\varepsilon)$ in $M \setminus \partial M$ is cpt.

We know a lot about $H_i(M \setminus \partial M, \partial M \times (0,\varepsilon); R)$ and this will help us compute $H_i(M, \partial M; R)$.

Prop. M n-mf w/ bdy, $\partial M \neq \emptyset$. Suppose M is cpt. Then

1) $H_i(M, \partial M; R) = 0$ if $i > n$.

2) $\exists [M, \partial M] \in H_n(M, \partial M; R)$ if M is R-orientable, such that $\forall x \in M \setminus \partial M$:

$$\begin{array}{ccccc} H_n(M, \partial M; R) & \longrightarrow & H_n(M, M\setminus\{x\}; R) & \underset{exc}{\cong} & H_n(M\setminus\partial M, M\setminus\partial M\setminus\{x\}; R) \\ [M,\partial M] & \longmapsto & & & \mu_x \end{array}$$

μ_x orientation classes for $M \setminus \partial M$

Pf: Lemma about $H_*(N, N\setminus K)$ for K cpt.: $\partial M \times (0,\varepsilon)$ has a cpt complement in $M\setminus\partial M$

$$\Rightarrow H_i(M\setminus\partial M, \partial M \times (0,\varepsilon)) = 0 \quad \forall i > n$$
$$\cong H_i(M, \partial M)$$

For small ε:
and $x \notin \partial M \times (0,\varepsilon)$

$$\begin{tikzcd} H_n(M,\partial M) \arrow[r, "\cong"] \arrow[d] & H_n(M, \partial M \times [0,\varepsilon)) \arrow[r, "\cong"', "exc"] \arrow[d] & H_n(M\setminus\partial M, \partial M\times(0,\varepsilon)) \arrow[d] & \exists!\,[M,\partial M]_\varepsilon \arrow[d, mapsto] \\ H_n(M, M\setminus\{x\}) \arrow[r, "="] & H_n(M, M\setminus\{x\}) \arrow[r, "\cong"', "exc"] & H_n(M\setminus\partial M, M\setminus\partial M\setminus\{x\}) & \mu_x \end{tikzcd}$$

$\exists!\,[M,\partial M] \in H_n(M,\partial M)$

$[M,\partial M]$ is defined to be the element corresponding to $[M,\partial M]_\varepsilon$ under the iso in the first row.

NTC independence of ε: take $\varepsilon < \varepsilon'$.

$$\begin{array}{ccccccc} [M,\partial M] \in H_n(M,\partial M) & \cong & H_n(M, \partial M\times[0,\varepsilon)) & \cong & H_n(M\setminus\partial M, \partial M \times [0,\varepsilon)) & \ni & [M,\partial M]_\varepsilon \\ \| & & \downarrow & & \downarrow & & \downarrow \\ [M,\partial M] \in H_n(M,\partial M) & \cong & H_n(M, \partial M\times[0,\varepsilon')) & \cong & H_n(M\setminus\partial M, \partial M \times [0,\varepsilon')) & \ni & [M,\partial M]_{\varepsilon'} \end{array}$$

Lemma $\Rightarrow$ $[M,\partial M]_\varepsilon \longmapsto [M,\partial M]_{\varepsilon'}$. Commutativity $\Rightarrow$ the classes $[M,\partial M]$ defined by ε resp. ε' agree.

$\forall x \in M\setminus\partial M$ $\exists \varepsilon > 0$ s.t. $x \notin \partial M \times [0,\varepsilon)$, so this construction indeed works.

Def. $[M,\partial M]$ is called the orientation class.

Lemma. M cpt R-orientable n-mf w/ bry. Then

1) $H_*(M;R) \cong H_*(M\setminus\partial M;R)$ (This is like removing out a measure zero subset in measure theory.)

2) $H^\ell(M,\partial M;R) \cong H_c^\ell(M\setminus\partial M;R)$

Pf: 1) Enough to show $H_*(M\setminus\partial M)=0$ and use the LES.

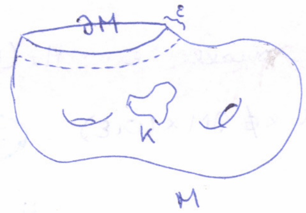
$$H_*(M, M\setminus\partial M) \underset{exc}{\cong} H_*(\partial M\times[0,1), \partial M\times(0,1)) = 0$$

since $\partial M\times(0,1) \hookrightarrow \partial M\times[0,1)$ is a homotopy equivalence. (not a def. retr.)

2) $H_c^\ell(M\setminus\partial M) \cong \underset{\substack{K\subseteq M\setminus\partial M\\ \text{compact}}}{\operatorname{colim}} H^\ell(M\setminus\partial M, M\setminus\partial M\setminus K)$

any K is included in $(M\setminus\partial M)\setminus(\partial M\times[0,\varepsilon])$ for small enough $\varepsilon>0$

cofinality $\cong \underset{\varepsilon\to0}{\operatorname{colim}} H^\ell(M\setminus\partial M, \partial M\times[0,\varepsilon))$



$\cong H^\ell(M\setminus\partial M, \partial M\times[0,\varepsilon_0))$

for small enough ε_0

This follows from the exercise in which every inclusion map for the colimit was an iso.

$\underset{exc.}{\cong} H^\ell(M,\partial M)$ □

Ex. Let M be the Möbius map, $\partial M\cong S^1$. M is not orientable:

In general, if M is orientable, $H_n(M,\partial M)\neq0$ since there is a nontrivial map $H_n(M,\partial M)\longrightarrow H_n(M,M\setminus\{x\})\cong\mathbb{Z}$ by the above discussion.

Back to the example: use the LES

$$\underbrace{H_2(M)}_{0 \text{ since } M\simeq S^1} \longrightarrow H_2(M,\partial M) \longrightarrow \underbrace{H_1(\partial M)}_{\cong\mathbb{Z}} \longrightarrow \underbrace{H_1(M)}_{\cong\mathbb{Z} \text{ since } M\simeq S^1}$$

$M \overset{r}{\rightleftarrows} S^1$ def retraction where S^1 is the middle circle.

Then $S^1\cong\partial M \hookrightarrow M \xrightarrow{r} S^1$ is a degree 2 map. $\Rightarrow H_2(M,\partial M)=0$.

By the remark above, this disproves orientability.

Ex. $C :=$ cylinder. $C \simeq S^1$, $\partial C \cong S^1 \amalg S^1$

$$\underbrace{H_2(C)}_{0} \longrightarrow H_2(C, \partial C) \longrightarrow \underbrace{H_1(\partial C)}_{\mathbb{Z}\oplus\mathbb{Z}} \xrightarrow{\nabla} \underbrace{H_1(C)}_{\mathbb{Z}}$$

∇ is the fold map: $\nabla(x,y) := x+y$

$\Rightarrow$ $H_2(C,\partial C) \cong \mathbb{Z}$ and the map $H_2(C,\partial C) \longrightarrow H_1(\partial C)$ is injective.

Now let $R = \mathbb{Z}$ or $R = \mathbb{Z}/2\mathbb{Z} = \mathbb{F}_2$, M an R-ori n-mf, cpt.

Prop. (Lefschetz duality) $H^k(M, \partial M; R) \cong H_{n-k}(M; R)$.

Pf: $H^k(M,\partial M) \cong H^k_c(M \setminus \partial M) \underset{PD}{\cong} H^{lf}_{n-k}(M\setminus \partial M) \cong H_{n-k}(M)$

$H^n(M, \partial M; \mathbb{Z}) \cong H_0(M;\mathbb{Z})$ free abelian non-trivial

UCT: $0 \longrightarrow \underbrace{\mathrm{Ext}^1_{\mathbb{Z}}(\underbrace{H_{n-1}(M,\partial M)}_{\text{torsion}}, \mathbb{Z})}_{0} \longrightarrow \underbrace{H^n(M,\partial M)}_{\text{free ab.}} \xrightarrow{\cong} \mathrm{Hom}(H_n(M,\partial M), \mathbb{Z}) \longrightarrow 0$

$\mathrm{Torsion}(H_{n-1}(M,\partial M)) = 0$, $H^{n+1}(M,\partial M) = 0$

UCT $\Rightarrow$ $\mathrm{Ext}^1_{\mathbb{Z}}(H_n(M,\partial M), \mathbb{Z}) = 0$ $\Rightarrow$ $\mathrm{Torsion}(H_n(M,\partial M)) = 0$ $\Rightarrow$ $H_n(M,\partial M)$ is free ab.

If M is connected: $H^n(M,\partial M) \cong H_0(M) \cong \mathbb{Z}$ $\Rightarrow$ $\mathrm{Hom}(H_n(M,\partial M), \mathbb{Z}) \cong \mathbb{Z}$

$\Rightarrow$ $H_n(M,\partial M) \cong \mathbb{Z}$.

Lemma. (PO) Suppose N to be an n-mf., $Y \subseteq N$ open subset, $N \setminus Y$ non-cpt and connected. $\Rightarrow$ $H_n(N, Y, G) = 0$.

Prop. Suppose M to be a compact n-mf. w/ bdry, orientable. Then ∂M is a cpt $(n-1)$-mf., orientable. Moreover the connecting homomorphism

$$\begin{array}{ccc} H_n(M,\partial M) & \xrightarrow{\partial} & H_{n-1}(\partial M) \\ {[M,\partial M]} & \longmapsto & [\partial M] = \partial[M,\partial M] \end{array}$$

sends the fund. class to a fund. class.

Pf (SKETCH): Suppose M to be connected, $\partial M \neq \emptyset$, ∂M is connected.

$$\begin{array}{ccccccc} H_n(M) & \longrightarrow & H_n(M, \partial M) & \longrightarrow & H_{n-1}(\partial M) & \longrightarrow & H_{n-1}(M) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & & & \\ H_n(M \setminus \partial M) = 0 & & \mathbb{Z} \neq 0 & & & & \end{array}$$

$\Rightarrow H_{n-1}(\partial M; \mathbb{Z}) \neq 0$

$$\left.\begin{array}{l} \Rightarrow H_{n-1}(\partial M; \mathbb{Z}) \neq 0 \\ \text{Trivial: } \partial M \text{ is an } (n-1)\text{-mf.} \end{array}\right\} \Rightarrow \text{orientability of } \partial M$$

$$0 \longrightarrow \underset{\mathbb{Z}}{\overset{}{H_n(M, \partial M)}} \xrightarrow{\partial} \underset{\mathbb{Z}}{H_{n-1}(\partial M)} \longrightarrow \begin{array}{c}\text{torsion}\\\text{free}\end{array} \longrightarrow 0$$

$H_{n-1}(M) \cong H^1(M, \partial M; \mathbb{Z}) \underset{UCT}{\cong} \mathrm{Hom}(H_1(M, \partial M), \mathbb{Z})$, $H_{n-1}(M)$ is torsion free.

$\Rightarrow \partial$ is an iso.

If ∂M is not connected, $\partial M = A \sqcup B$, A path-connected, $B \neq \emptyset$:

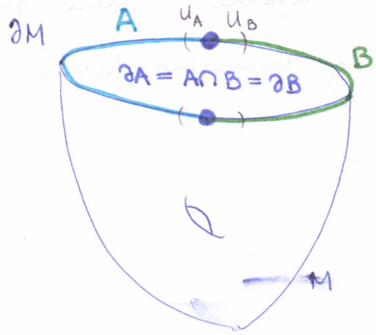
$$\begin{tikzcd} H_n(M, \partial M) \arrow[r, "\partial"] \arrow[drr, "\cong"'] & H_{n-1}(\partial M) \arrow[r] & H_{n-1}(\partial M, B) \arrow[d, "\cong"] \\ & & H_{n-1}(A) \end{tikzcd}$$

This is done using the Lemma above on non-cpt complement and $H_*(X, \mathbb{Q}/\mathbb{Z})$. (PO) □

Thm. (Poincaré — Lefschetz duality) M cpt n-dimensional oriented manifold with boundary. Suppose $\partial M = A \cup B$ where A and B are $(n-1)$-manifolds with boundary being common, i.e. $\partial A = \partial B = A \cap B$.

Then $H^\ell(M, B) \xrightarrow{[M, \partial M] \cap -} H_{n-\ell}(M, B)$ is an isomorphism.

Note that $B = \emptyset$ has been already covered: Lefschetz duality.



$A \setminus \partial A$ and $B \setminus \partial B$ are open in M

Pf (SKETCH): Case 1 $B = \emptyset$

$$\begin{tikzcd}
H^{\ell}(M, \partial M) \arrow[r, "{[M,\partial M]\cap -}"] & H_{n-\ell}(M) \\
H^{\ell}_c(M\setminus\partial M) \arrow[u, "\cong"] \arrow[r] \arrow[ur, phantom, "G"] & H_{n-\ell}(M\setminus\partial M) \arrow[u, "\cong"]
\end{tikzcd}$$

This follows from everything we did today.

Case 2. General case; want to reduce to Case 1.
$\{A,B\}$ is an excisive couple, i.e. $H_*(M,\partial A) \cong H_*(B,\partial B)$, as we will now show.
We can find U_A, U_B collar nbhoods, $A\cap B \subseteq U_A \subseteq A$, $A \cap B \subseteq U_B \subseteq B$ for $\partial A = A\cap B = \partial B$.

$U := (B \cup U_A) \times [0,1) \subseteq \partial M \times [0,1)$

$V := (A \cup U_B) \times [0,1) \subseteq \partial M \times [0,1)$

$U \cup V = \partial M \times [0,1) \simeq \partial M$

$U \cap V \simeq A \cap B$, $U \simeq B$, $V \simeq A$ $\Rightarrow$ $\{A,B\}$ is excisive. ✓

$$\begin{tikzcd}
\cdots \arrow[r] & H^{\ell}(M,\partial M) \arrow[r] \arrow[d, "\cong", "{[M,\partial M]\cap -}"', "\text{Case 1}"] & H^{\ell}(M,A) \arrow[r] \arrow[d, "{[M,\partial M]\cap -}"] & H^{\ell}(\partial M, A) \arrow[r] \arrow[d, "\cong\,\text{Case 1}", "{H^{\ell}(B,\partial B)\ [B,\partial B]\cap -}"'] & H^{\ell+1}(M,\partial M) \arrow[r] \arrow[d, "{[M,\partial M]\cap -}"] & \cdots \\
\cdots \arrow[r] & H_{n-\ell}(M) \arrow[r] & H_{n-\ell}(M,B) \arrow[r, "\partial"] & H_{n-\ell-1}(B) \arrow[r] & H_{n-\ell-1}(M) \arrow[r] & \cdots
\end{tikzcd}$$

here we use that $\{A,B\}$ is excisive

We have seen a similar diagram: painful but not difficult to see commutativity.

5-lemma $\Rightarrow$ PL duality.

Alexander duality: $K \subseteq S^n$ compact, proper non-empty, locally contractible subset

$$\Rightarrow \tilde{H}_{\ell}(S^n\setminus K; \mathbb{Z}) \cong \tilde{H}^{n-\ell-1}(K;\mathbb{Z})$$

$$\tilde{H}_{\ell}(S^n\setminus K; \mathbb{Z}) \overset{PD}{=} H^{\ell}_c(S^n \setminus K) \quad \text{for } \ell \neq 0$$

The interesting thing is that only the shape of K matters, not how it is embedded into S^n.

More generally: $\tilde{H}_\ell(S^n \setminus K;\mathbb{Z}) \cong \check{H}^{n-\ell-1}(K;\mathbb{Z})$ for Čech cohomology.